

# Hořava-Lifshitz Gravity And Ghost Condensation

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**ABSTRACT:** In this paper we formulate RFDiff invariant  $f(R)$  Hořava-Lifshitz gravity and we show that it is related to the ghost condensation in the projectable version of Hořava-Lifshitz gravity.

**KEYWORDS:** Hořava-Lifshitz gravity, Ghost Condensation.

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## 1. Introduction and Summary

Recently Petr Hořava formulated idea considering consistent renormalizable quantum theory of gravity [1] (see also [2, 3, 4, 5]). This proposal is based on an idea that the ultra-violet (UV) behavior of quantum gravity is improved thanks to terms with higher spatial derivatives where at the same time the number of time derivatives in the Lagrangian remains equal to two so that there is no problem with ghosts that arise in Lorentz invariant higher-derivative theories of gravity. It is clear that the breaking the symmetry between space and time we lose the Lorentz Invariance of given theory that now is not the fundamental symmetry of the theory and can emerge at low energies as an approximative symmetry.

However it turned out that even if the Hořava's formulation is very promising and interesting there are many conceptional problems that arise in this theory as in any theory of gravity with reduced diffeomorphism group. Explicitly, the fact that the theory is not invariant under full diffeomorphism group implies an existence of new degrees of freedom which is the mode of helicity zero. This mode has natural geometrical origin since the introduction of the preferred time coordinate leads to the foliation of the space-time manifold by space-like surfaces where the helicity-0 mode is excitation of this foliation structure. Very interesting analysis of the properties of this mode has been done recently in [6]. The main conclusion derived here is that this extra mode does not decouple at low energies and hence it is questionable whether Hořava-Lifshitz gravity can flow to General Relativity at low energies. On the other hand authors in [6] suggested very interesting possibility that Hořava-Lifshitz gravity could flow at low energies to Lorentz violating model of modified gravity where the modifications are small so that they do not contradict to experimental data. On the other hand the analysis of the modified gravity models performed in the past shown that properties of these extra modes imply that these modified gravity modes are not phenomenologically acceptable <sup>1</sup>. Unfortunately this situation occurs in the original version of Hořava's proposal where the extra mode possesses pathological behavior [10, 11, 12].

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<sup>1</sup>For review, see [7, 8, 9].

In [6] three different models of Hořava-Lifshitz gravities were studied. The first one is the projectable version of the original proposal [1] where the lapse function depends on time only. The second one is the model with smaller symmetry group (Denoted as RFDiff invariant theory in [6]) and can be considered as a power counting renormalizable version of the ghost condensation [13]. However as was nicely shown in [6] the spectrum of this model possesses the second helicity-0 mode that unfortunately leads to fast instabilities or to the break-down of the perturbative description. These facts imply that it is unclear whether this model can be a promising candidate for description of quantum gravity. Finally the third model studied in [6] is the so called "healthy-extended" Hořava-Lifshitz gravity introduced in [14, 15]<sup>2</sup>. It was shown there that now the scalar sector does not suffer from pathologies and that this model is compatible with phenomenological constraints for suitable choices of parameters. Hence this model can be considered as a starting point for constructing a renormalizable theory of quantum gravity.

In this paper we consider some aspects of RFDiff invariant Hořava-Lifshitz theories. Namely we present construction of these models based on the original Hořava's "detailed balance condition" [1] that was introduced in [18, 19]. This construction leads to models that are invariant under transformations

$$t' = t + \delta t, \quad x'^i = x^i + \xi^i(\mathbf{x}), \quad (1.1)$$

where  $\delta t = \text{const}$  and where  $\xi^i$  are space dependent parameters of spatial diffeomorphism. Then we extend symmetries of given model when we demand that the action should be invariant under transformation

$$t' = t' = t + \delta t, \quad x'^i = x^i + \xi^i(\mathbf{x}, t). \quad (1.2)$$

that is exactly the symmetry group of RFDiff invariant theories. It is important that this symmetry group is different from *foliation preserving diffeomorphism* that has the form

$$t' = f(t), \quad x'^i = x^i + \xi^i(\mathbf{x}, t). \quad (1.3)$$

In order to construct action that is invariant under (1.2) we have to introduce the fields  $N^i$  that are well known "shifts" from 3+1 analysis of General Relativity. Note that we do not need to introduce the lapse function  $N$ . As a consequence of this fact the Hamiltonian constraint is absent in theories invariant under (1.2). In other words the Hamiltonian of RFDiff invariant Hořava-Lifshitz theory has the structure of the Hamiltonian of diffeomorphism invariant theory where the temporal diffeomorphism has been fixed. In fact, we explicitly construct such a theory and we argue that RFDiff invariant Hořava-Lifshitz theory can be interpreted as a result of the ghost condensation in Hořava-Lifshitz gravity coupled to specific form of the scalar field action. We also argue that in order to have the first class constraint that can be fixed we should consider projectable version of Hořava-Lifshitz gravity. In fact, it was shown in [20] that the Hamiltonian constraint in non-projectable Hořava-Lifshitz gravity is the second class constraint and it certainly does not make sense

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<sup>2</sup>For Hamiltonian analysis of this model, see [16, 17].

to fix it. On the other hand the projectable version of Hořava-Lifshitz gravity is characterized by the global form of the Hamiltonian constraint that is trivially the first class constraint and hence the gauge fixing can be performed.

Let us outline our results. We construct general form of RFDiff invariant  $f(R)$  Hořava-Lifshitz gravity theories that are generalizations of the theories studied in [6]. We show that they can be derived by ghost condensation from the projectable version of  $f(R)$  Hořava-Lifshitz gravities.

The structure of this paper is as follows. In the next section (2) we define  $f(R)$  models of gravity that obey the detailed balance condition and that are invariant under symmetry group (1.1). Then in section (3) we extend symmetries of given theories so that they are invariant under (1.2) and hence they are RFDiff-invariant  $f(R)$  Hořava-Lifshitz gravities. Finally in section (4) we show that these theories can be interpreted as the ghost condensation in the projectable version of Hořava-Lifshitz gravity.

## 2. $f(R)$ Gravity at Criticality

In this section we introduce models of gravity based on original Hořava's proposal [3] and its generalization for the the construction of  $f(R)$  Hořava-Lifshitz gravities performed in [18, 19] that are invariant under (1.1) <sup>3</sup>. Following [3] we assume an existence of  $D + 1$  dimensional quantum theory of gravity that is characterized by following quantum Hamiltonian density

$$\hat{\mathcal{H}} = \kappa^2 \sqrt{\hat{g}} \left( \sum_{n=0}^{\infty} \hat{c}_n(\hat{g}_{ij}) \left( \hat{Q}^{\dagger ij} \frac{1}{\hat{g}} \hat{\mathcal{G}}_{ijkl} \hat{Q}^{kl} \right)^n - \hat{c}_0(\hat{g}_{ij}) \right) , \quad (2.1)$$

where

$$\hat{Q}^{\dagger ij} = -i\hat{\pi}^{ij} + \sqrt{\hat{g}}\hat{E}^{ij}(\hat{g}_{ij}) , \quad \hat{Q}^{ij} = i\hat{\pi}^{ij} + \sqrt{\hat{g}}\hat{E}^{ij}(\hat{g}_{ij}) , \quad (2.2)$$

and where  $\hat{g} = \det \hat{g}_{ij}$  and  $\kappa$  is a coupling constant of given theory. Note that the fundamental operators of quantum theory of gravity are metric components  $\hat{g}_{ij}(\mathbf{x})$ ,  $i = 1, \dots, D$ ,  $\mathbf{x} = (x^1, \dots, x^D)$  together with their conjugate momenta  $\hat{\pi}^{ij}(\mathbf{x})$ . These operators obey the commutation relations

$$[\hat{g}_{ij}(\mathbf{x}), \hat{\pi}^{kl}(\mathbf{y})] = \frac{1}{2}(\delta_i^k \delta_j^l + \delta_i^l \delta_j^k) \delta(\mathbf{x} - \mathbf{y}) . \quad (2.3)$$

Further,  $\hat{c}_n$  defined in (2.1) are scalar functions that depend on  $\hat{g}_{ij}$  only. It is clear that in the Schrödinger representation the operators (2.2) take the form

$$\hat{Q}^{ij}(\mathbf{x}) = -\frac{\delta}{\delta g^{ij}(\mathbf{x})} + \sqrt{g}(\mathbf{x}) E^{ij}(\mathbf{x}) , \quad \hat{Q}^{\dagger ij}(\mathbf{x}) = \frac{\delta}{\delta g^{ij}(\mathbf{x})} + \sqrt{g}(\mathbf{x}) E^{ij}(\mathbf{x}) . \quad (2.4)$$

The next goal is to specify the form of the operators  $E^{ij}$ . To do this we assume that the theory obeys the *detailed balance condition* so that

$$\sqrt{g}(\mathbf{x}) E^{ij}(\mathbf{x}) = \frac{1}{2} \frac{\delta W}{\delta g^{ij}(\mathbf{x})} , \quad (2.5)$$

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<sup>3</sup> $f(R)$  Hořava-Lifshitz gravities were also extensively studied in [30, 31, 32, 32].

where  $W$  is an action of  $D$  dimensional gravity. As in [3] we construct the vacuum wave functional of  $D + 1$  dimensional theory as

$$\Psi[g(\mathbf{x})] = \exp\left(-\frac{1}{2}W\right) , \quad (2.6)$$

where  $W$  is the Einstein-Hilbert action in  $D$  dimensions

$$W = \frac{1}{2\kappa_W^2} \int d^D \mathbf{x} \sqrt{g} R . \quad (2.7)$$

Generally the action  $W$  could also contains additional terms that are functions of metric however the explicit form of  $W$  will not be important in following discussion.

The form of the vacuum wave functional (2.6) implies that it is annihilated by (2.1). Further as a consequence of the detailed balance condition the norm of the functional (2.6) coincides with the partition function of  $D$  dimensional Euclidean gravity. In other words we have again infinite number of possible Hamiltonians that annihilate the vacuum state (2.6) and that are defined using the principle of detailed balance.

In order to find the Lagrangian formulation of this theory we now consider the classical form of the Hamiltonian density (2.1) that we can now write in the form

$$\mathcal{H}(t, \mathbf{x}) = \kappa^2 \sqrt{g} f \left( Q^{\dagger ij} \frac{1}{g} \mathcal{G}_{ijkl} Q^{kl} \right) , \quad (2.8)$$

where  $f$  is an arbitrary function that can be defined by its Taylor expansion as in (2.1). Further, the functions  $Q^{ij}$  and  $Q^{\dagger ij}$  are defined as

$$Q^{ij} = i\pi^{ij} + \sqrt{g} E^{ij} , \quad Q^{\dagger ij} = -i\pi^{ij} + \sqrt{g} E^{ij} , \quad (2.9)$$

where  $g_{ij}$ ,  $i, j = 1, \dots, D$  are components of metric and  $\pi^{ij}$  are conjugate momenta. These canonical variables have non-zero Poisson brackets

$$\left\{ g_{ij}(\mathbf{x}), \pi^{kl}(\mathbf{y}) \right\} = \frac{1}{2} (\delta_i^k \delta_j^l + \delta_i^l \delta_j^k) \delta(\mathbf{x} - \mathbf{y}) . \quad (2.10)$$

Finally  $\mathcal{G}_{ijkl}$  denotes the inverse of the De Witt metric

$$\mathcal{G}_{ijkl} = \frac{1}{2} (g_{ik} g_{jl} + g_{il} g_{jk}) - \tilde{\lambda} g_{ij} g_{kl} \quad (2.11)$$

with  $\tilde{\lambda} = \frac{\lambda}{D\lambda - 1}$ . The "metric on the space of metric",  $\mathcal{G}^{ijkl}$  is defined as

$$\mathcal{G}^{ijkl} = \frac{1}{2} (g^{ik} g^{jl} + g^{il} g^{jk} - \lambda g^{ij} g^{kl}) \quad (2.12)$$

with  $\lambda$  an arbitrary real constant. Note that (2.11) together with (2.12) obey the relation<sup>4</sup>

$$\mathcal{G}_{ijmn} \mathcal{G}^{mnkl} = \frac{1}{2} (\delta_i^k \delta_j^l + \delta_i^l \delta_j^k) . \quad (2.13)$$

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<sup>4</sup>Note that we use the terminology introduced in [3] and that we review there. In case of relativistic theory, the full diffeomorphism invariance fixes the value of  $\lambda$  uniquely to equal  $\lambda = 1$ . In this case the object  $\mathcal{G}_{ijkl}$  is known as the "De Witt metric". We use this terminology to more general case when  $\lambda$  is not necessarily equal to one.

Using (2.8) we now determine corresponding Lagrangian. We begin with the canonical equation of motion for  $g_{ij}$

$$\partial_t g_{ij} = \{g_{ij}, H\} = 2\kappa^2 \frac{1}{\sqrt{g}} \mathcal{G}_{ijkl} \pi^{kl} f' \left( Q^{\dagger ij} \frac{1}{g} \mathcal{G}_{ijkl} Q^{kl} \right) , \quad (2.14)$$

where  $H = \int d^D \mathbf{x} \mathcal{H}$  with  $\mathcal{H}$  given in (2.8) and where  $f'(x) = \frac{df}{dx}$ . Using this equation we find

$$\partial_t g_{ij} \mathcal{G}^{ijkl} \partial_t g_{kl} = 4\kappa^4 \left( \pi^{ij} \frac{1}{g} \mathcal{G}_{ijkl} \pi^{kl} \right) f'^2 \left( \pi^{ij} \frac{1}{g} \mathcal{G}_{ijkl} \pi^{kl} + E^{ij} \mathcal{G}_{ijkl} E^{kl} \right) . \quad (2.15)$$

For further purposes we introduce the notation

$$G = \frac{1}{4\kappa^4} \partial_t g_{ij} \mathcal{G}^{ijkl} \partial_t g_{kl} , \quad V(g) = E^{ij} \mathcal{G}_{ijkl} E^{kl} , \quad P = \pi^{ij} \frac{1}{g} \mathcal{G}_{ijkl} \pi^{kl} . \quad (2.16)$$

At this place we would like to stress that generally we can abandon the detailed balance condition and consider  $V(g)$  as general potential for the metric. Then we can presume that (2.15) can be solved for  $P$  as

$$P = \Psi(G, V) \quad (2.17)$$

so that  $\Psi$  obeys the equation

$$G = \Psi f'^2 (\Psi + V) . \quad (2.18)$$

Taking derivative of this equation with respect to  $G$  we find the useful relation

$$1 = \frac{d\Psi}{dG} f'^2 + 2\Psi f' f'' \frac{d\Psi}{dG} . \quad (2.19)$$

Then it is easy to see that corresponding Lagrangian takes the form

$$\begin{aligned} L &= \int d^D \mathbf{x} \mathcal{L} = \int d^D \mathbf{x} (\pi^{ij} \partial_t g_{ij} - \mathcal{H}) = \\ &= \kappa^2 \int d^D \mathbf{x} \sqrt{g} (2\Psi(G, V) f'(\Psi(G, V)) - f(\Psi(G, V))) . \end{aligned} \quad (2.20)$$

As the next step we show that the action

$$S = \int dt d^D \mathbf{x} \mathcal{L} , \quad (2.21)$$

where  $\mathcal{L}$  is given in (2.20) is invariant under the transformation

$$t' = t + \delta t , \delta t = \text{const} , \quad x'^i = x^i(\mathbf{x}) . \quad (2.22)$$

This follows from the fact that we presumed that the functional  $W$  is invariant under the spatial diffeomorphism under which the metric  $g_{ij}$  and tensor  $E^{ij}$  transform as

$$\begin{aligned} g'_{ij}(\mathbf{x}') &= g_{kl}(\mathbf{x}) (D^{-1})^k_i (D^{-1})^l_j , \\ E'^{ij}(\mathbf{x}') &= E^{kl}(\mathbf{x}) D^i_k D^j_l , \end{aligned} \quad (2.23)$$

where

$$D_j^i = \frac{\partial x'^i}{\partial x^j} , \quad D_j^i (D^{-1})_k^j = \delta_k^i . \quad (2.24)$$

Using the transformation property of  $g_{ij}$  we find that the metric  $\mathcal{G}_{ijkl}$  transforms as

$$\mathcal{G}'_{ijkl}(\mathbf{x}') = \mathcal{G}_{i'j'k'l'}(\mathbf{x}) (D^{-1})_i^{i'} (D^{-1})_j^{j'} (D^{-1})_k^{k'} (D^{-1})_l^{l'} . \quad (2.25)$$

Finally, using the fact that  $d^D \mathbf{x}' \sqrt{g'(\mathbf{x}')} = d^D \mathbf{x} \sqrt{g(\mathbf{x})}$  we see that the invariance of the action under the spatial diffeomorphism (2.22) is obvious.

### 3. Extension of Symmetries

We argued that the Lagrangian (2.20) is invariant under  $D$ -dimensional *time independent spatial diffeomorphism*. In [1, 3] these symmetries were extended to the diffeomorphism that respect the preferred codimension-one foliation  $\mathcal{F}$  of the theory by the slices of fixed time. On the other hand we make following extension of symmetries

$$t' = t + \delta t , \quad \delta t = \text{const} , \quad x'^i = x^i + \xi^i(\mathbf{x}, t) \quad (3.1)$$

that is RFDiffs symmetry group in terminology of [6]. Let us now study consequences of the requirement of the invariance of the action under (3.1).

We firstly note that the metric components  $g_{ij}$  transform under (3.1) as

$$\begin{aligned} g'_{ij}(\mathbf{x}', t') &= g_{ij}(\mathbf{x}, t) - \partial_i \xi^k(t, \mathbf{x}) g_{kj}(\mathbf{x}, t) - g_{ik}(t, \mathbf{x}) \partial_j \xi^k(\mathbf{x}, t) , \\ g'^{ij}(\mathbf{x}', t') &= g^{ij}(\mathbf{x}, t) + \partial_k \xi^i(t, \mathbf{x}) g^{kj}(\mathbf{x}, t) + g^{ik}(t, \mathbf{x}) \partial_k \xi^j(\mathbf{x}, t) . \end{aligned} \quad (3.2)$$

However now due to the fact that the gauge parameter  $\xi^i$  depends on time we find that  $\partial_t g_{ij}$  does not transform covariantly under (3.1). In order to find an action that is invariant under (3.1) it is necessary to introduce new fields  $N_i(t, \mathbf{x})$  that transform under (3.1) as

$$\begin{aligned} N'_i(\mathbf{x}', t') &= N_i(\mathbf{x}, t) - g_{ij}(\mathbf{x}, t) \partial_t \xi^j(\mathbf{x}, t) - \partial_i \xi^j(\mathbf{x}, t) N_j(\mathbf{x}, t) , \\ N^i(\mathbf{x}', t') &= N^i(\mathbf{x}, t) - \partial_t \xi^i(t, \mathbf{x}) + N^j(t, \mathbf{x}) \partial_j \xi^i(t, \mathbf{x}) . \end{aligned} \quad (3.3)$$

Let us define

$$\hat{K}_{ij} = \partial_t g_{ij} - D_i N_j - D_j N_i , \quad (3.4)$$

where  $D_i$  is a covariant derivative constructed from  $g_{ij}$  that obeys  $D_k g_{ij} = 0$ . Then it is easy to see that

$$\hat{K}'_{ij}(\mathbf{x}', t') = \hat{K}_{ij}(t, \mathbf{x}) - \partial_i \xi^k(t, \mathbf{x}) \hat{K}_{kj}(t, \mathbf{x}) - \hat{K}_{ik}(t, \mathbf{x}) \partial_j \xi^k(t, \mathbf{x}) \quad (3.5)$$

that means that  $\hat{K}_{ij}$  transforms covariantly under (3.1). Hence the natural generalization of (2.21) takes the form

$$S = \int dt d^D \mathbf{x} \mathcal{L} , \quad \mathcal{L} = \kappa^2 \sqrt{g} (2\Psi(\hat{G}, V) f'(\Psi(\hat{G}, V)) - f(\Psi(\hat{G}, V))) , \quad (3.6)$$

where

$$\hat{G} \equiv \frac{1}{4\kappa^2} \hat{K}_{ij} \mathcal{G}^{ijkl} \hat{K}_{kl} . \quad (3.7)$$

This Lagrangian can be considered as generalization of the RFDiff-invariant theories studied in [6].

It is instructive to determine Hamiltonian from the action (3.6). We firstly find canonical momenta from (3.6)

$$\begin{aligned} \pi^{ij} &= \frac{\delta S}{\delta \partial_t g_{ij}} = \frac{1}{2\kappa^2} \sqrt{g} \mathcal{G}^{ijkl} \pi_{kl} \left( 2 \frac{d\Psi}{d\hat{G}} f' + 2\Psi \frac{d\Psi}{d\hat{G}} f'' - f' \frac{d\Psi}{d\hat{G}} \right) = \\ &= \frac{1}{2\kappa^2} \sqrt{g} \mathcal{G}^{ijkl} \hat{K}_{kl} \frac{1}{f'} , \quad \pi^i = \frac{\delta S}{\delta \partial_t N_i} \approx 0 , \end{aligned} \quad (3.8)$$

where we used (2.19). Using this relation we can easily find corresponding Hamiltonian

$$\begin{aligned} H &= \int d^D \mathbf{x} (\partial_t g_{ij} \pi^{ij} - L) = \\ &= \int d^D \mathbf{x} (2\kappa^2 \sqrt{g} f + N^i \mathcal{H}_i) , \end{aligned} \quad (3.9)$$

where

$$\mathcal{H}_i = -g_{ik} D_j \pi^{jk} \approx 0 \quad (3.10)$$

is standard secondary constraint related to the primary constraint  $\pi^i \approx 0$ . In other words this theory is invariant under spatial diffeomorphism generated by

$$\mathbf{T}_S(N^i) = \int d^D \mathbf{x} N^i \mathcal{H}_i . \quad (3.11)$$

On other hand we see from the structure of the Hamiltonian (3.9) that the Hamiltonian constraint is absent. In the next section we show that this Hamiltonian is related to specific form of ghost condensation.

#### 4. RFDiff invariant Hořava-Lifshitz Gravity and Ghost Condensation

We again begin with the Hamiltonian formulation of the Hořava-Lifshitz gravity where the Hamiltonian density (without the first class constraints that generate the spatial diffeomorphism) takes the form

$$\mathcal{H} = \kappa^2 \sqrt{g} f \left( \pi^{ij} \frac{1}{g} \mathcal{G}_{ijkl} \pi^{kl} + E^{ij} \mathcal{G}_{ijkl} E^{kl} \right) . \quad (4.1)$$



Now we presume that this Hamiltonian density arises in the process of the specific form of the gauge fixing. Explicitly we consider the system of the gravity coupled with scalar field. The dynamics of this system is governed by Hamiltonian that is the sum of the first class constraints

$$H^G = \int d^D \mathbf{x} \mathcal{H}_0(\mathbf{x}) + \int d^D \mathbf{x} N^i \mathcal{H}_i(\mathbf{x}) , \quad \mathcal{H}^i = -2g_{ik} \nabla_j \pi^{jk} + p_\phi \partial_i \phi . \quad (4.2)$$

The canonical variables for the scalar field are  $\phi$  and the momentum conjugate  $p_\phi$  with non-zero Poisson brackets

$$\{\phi(\mathbf{x}), p_\phi(\mathbf{y})\} = \delta(\mathbf{x} - \mathbf{y}) . \quad (4.3)$$

By presumption  $H_0 = \int d^D \mathbf{x} N \mathcal{H}_0(\mathbf{x}) \approx 0$  is the first class constraint so that the Hamiltonian  $H^G$  weakly vanishes. We further presume that the gauge freedom generated by  $H_0$  can be fixed by the gauge fixing condition

$$\mathcal{G} = \phi(\mathbf{x}) - t = 0 . \quad (4.4)$$

In other words, we presume that  $\mathcal{G}$  together with  $H_0$  are the second class constraints that vanish strongly. As a result we find that the action of the gauge fixed theory takes the form

$$S = \int dt d^D \mathbf{x} (\pi^{ij} \partial_t g_{ij} + p_\phi \partial_t \phi - H^G) = \int dt d^D \mathbf{x} (\pi^{ij} \partial_t g_{ij} + p_\phi - N^i \mathcal{H}_i) . \quad (4.5)$$

From (4.5) we see that it is natural to identify the Hamiltonian of the gauge fixed theory with  $-p_\phi$ . On the other hand since we know that the Hamiltonian density of the gauge fixed theory is  $\mathcal{H}$  we have following identification

$$p_\phi = -\mathcal{H} \quad (4.6)$$

or equivalently, using (4.1) we can rewrite this relation into the form

$$\frac{1}{\kappa^4 g} p_\phi^2 = f^2 \left( \pi^{ij} \frac{1}{g} \mathcal{G}_{ijkl} \pi^{kl} + E^{ij} \mathcal{G}_{ijkl} E^{kl} \right) . \quad (4.7)$$

Now we presume that  $f^2$  has an inverse function that we denote as  $\Psi$ . Then we find

$$-\Psi \left( \frac{1}{\kappa^4 g} p_\phi^2 \right) + \pi^{ij} \frac{1}{g} \mathcal{G}_{ijkl} \pi^{kl} + E^{ij} \mathcal{G}_{ijkl} E^{kl} = 0 . \quad (4.8)$$

This equation can be interpreted as the strongly vanishing constraint  $\mathcal{H}_0$

$$\mathcal{H}_0 = -\kappa^2 \sqrt{g} \Psi \left( \frac{1}{\kappa^4 g} p_\phi^2 \right) + \kappa^2 \left( \pi^{ij} \frac{1}{\sqrt{g}} \mathcal{G}^{ijkl} \pi_{kl} + \sqrt{g} E^{ij} \mathcal{G}_{ijkl} E^{kl} \right) = 0 . \quad (4.9)$$

Clearly the Poisson bracket between  $\mathcal{H}_0$  and  $\mathcal{G}$  is non-zero that confirms that the constraints  $\mathcal{H}_0$  together with  $\mathcal{G}$  are the second class constraints.

Knowing the form of the Hamiltonian constraint  $\mathcal{H}_0$  we can find the Lagrangian density for the given system. As the first step we find the relation between  $\partial_t\phi$  and canonical variables

$$\partial_t\phi = \{\phi, H^G\} = -2N \frac{1}{\kappa^2 \sqrt{g}} p_\phi \Psi' . \quad (4.10)$$

The equation (4.10) implies

$$\frac{1}{4N^2} (\partial_t\phi)^2 = \frac{p_\phi^2}{\kappa^4 g} \Psi'^2 \left( \frac{p_\phi^2}{\kappa^4 g} \right) . \quad (4.11)$$

Now we presume that this equation can be solved for  $\frac{1}{\kappa^4 g} p_\phi^2$  as

$$\frac{1}{\kappa^4 g} p_\phi^2 = \Phi \left( \frac{1}{4N^2} (\partial_t\phi)^2 \right) . \quad (4.12)$$

In other words this equation together with (4.11) implies

$$I = \Phi \Psi'^2(\Phi(I)) , \quad (4.13)$$

where  $I = \frac{1}{4N^2} (\partial_t\phi)^2$ . Taking derivative this equation with respect to  $I$  we find

$$1 = \frac{d\Phi}{dI} \Psi'^2 + 2\Phi \frac{d\Psi}{d\Phi} \frac{d^2\Psi}{d^2\Phi} \frac{d\Phi}{dI} \quad (4.14)$$

that will be useful below.

With the help of these results it is easy to find the Lagrangian in the form

$$\begin{aligned} L &= \int d^D \mathbf{x} (\partial_t\phi p + \partial_t g_{ij} \pi^{ij} - N \mathcal{H}_0 - N^i \mathcal{H}_i) = \\ &= \int d^D \mathbf{x} N \sqrt{g} \left( \frac{1}{\kappa^2} K_{ij} \mathcal{G}^{ijkl} K_{kl} - \kappa^2 E^{ij} \mathcal{G}_{ijkl} E^{kl} + \right. \\ &\quad \left. + \kappa^2 N \sqrt{g} \Psi(\Phi(I)) - 2\kappa^2 N \sqrt{g} \frac{1}{\Psi'(\Phi(I))} I \right) , \end{aligned} \quad (4.15)$$

where  $K_{ij} = \frac{1}{N} (\partial_t g_{ij} - D_i N_j - D_j N_i)$ . Since we presumed that this Lagrangian was derived in the process of the gauge fixing we easily find its general form when we perform the substitution

$$I = \frac{1}{4N^2} \partial_t\phi \partial_t\phi \rightarrow \hat{I} = \frac{1}{4} (\nabla_n \phi)^2 - \Omega (g^{ij} \partial_i \phi \partial_j \phi) , \quad (4.16)$$

where

$$\nabla_n \phi = \frac{1}{N} (\partial_t \phi - N^i \partial_i \phi) , \quad (4.17)$$

and where  $\Omega(x)$  is function that reflects the anisotropy of the space-time [22, 23, 24, 25]. Then the Lagrangian density takes the form

$$\begin{aligned} \mathcal{L} = N\sqrt{g} & \left( \frac{1}{\kappa^2} K_{ij} \mathcal{G}^{ijkl} K_{kl} - \kappa^2 E^{ij} \mathcal{G}_{ijkl} E^{kl} + \right. \\ & \left. + \kappa^2 \Psi \left( \Phi \left( \hat{I} \right) \right) - 2\kappa^2 \frac{1}{\Psi'(\Phi(\hat{I}))} \hat{I} \right) . \end{aligned} \quad (4.18)$$

In order to check the consistency of our analysis we proceed in reverse direction and determine Hamiltonian for  $\phi$  from (4.18). The momentum  $p_\phi$  conjugate to  $\phi$  takes the form

$$\begin{aligned} p_\phi &= \frac{1}{2} \kappa^2 \sqrt{g} \nabla_n \phi \left( \Psi' \frac{d\Phi}{d\hat{I}} + 2\Psi'' \Phi \frac{d\Phi}{d\hat{I}} - \frac{1}{\Psi'} \right) = \\ &= -\frac{1}{2} \kappa^2 \sqrt{g} \nabla_n \phi \frac{1}{\Psi'} , \end{aligned} \quad (4.19)$$

where  $\Psi' = \frac{d\Psi}{d\Phi}$  and where we used (4.14). Then the Hamiltonian for  $p_\phi$  takes the form

$$\begin{aligned} H^\phi &= \int d^D \mathbf{x} (N \mathcal{H}_0^\phi + N^i \mathcal{H}_i^\phi) , \\ \mathcal{H}_0^\phi &= -\kappa^2 \sqrt{g} \Psi \left( \frac{1}{\kappa^4 g} p_\phi^2 \right) - \frac{1}{2} \kappa^2 \sqrt{g} \frac{1}{\Psi'(\frac{1}{\kappa^2 g} p_\phi^2)} \Omega(g^{ij} \partial_i \phi \partial_j \phi) , \quad \mathcal{H}_i^\phi = p_\phi \partial_i \phi . \end{aligned} \quad (4.20)$$

We see that this Hamiltonian constraint  $\mathcal{H}_0^\phi$  coincides with the  $\phi$ -part of the Hamiltonian constraint (4.9) (after fixing the gauge  $\phi = t$ ) which justifies our approach.

In summary, we found the action of Hořava-Lifshitz gravity and scalar field that leads to RFDiff invariant theory through the ghost condensation. At this place we should stress one subtle point in our analysis. It is well known that the gauge fixing in the Hamiltonian framework corresponds in the imposing of the additional constraints (Gauge fixing functions) on the system with the first class constraints such that the Poisson brackets between gauge fixing functions and the original first class constraints is non-zero on the constraint surface. As a consequence the gauge fixing functions together with the original first class constraints become the second class constraints that strongly vanish and can be explicitly solved<sup>5</sup>. In other words the gauge fixing makes sense in case when the constraint is the first class. However it was shown in [20] that the Hamiltonian constraint  $\mathcal{H}_0$  of the Hořava-Lifshitz theory with space dependent lapse function is the second class constraint. On the other hand the projectable version of Hořava-Lifshitz theory is characterized the global form of the Hamiltonian constraint

$$\mathbf{H} = \int d^D \mathbf{x} \mathcal{H}_0 . \quad (4.21)$$

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<sup>5</sup>For review, see [26, 27, 28].

Clearly  $\{\mathbf{H}, \mathbf{H}\} = 0$  and consequently  $\mathbf{H}$  can be considered as the first class constraint. Further the Poisson bracket between  $\mathbf{H}$  and  $\mathcal{G}(\mathbf{x})$  is non-zero and hence  $\mathbf{H}$  together with  $\mathcal{G}(\mathbf{x})$  form the second class constraints. The equation  $\mathbf{H} = 0$  can be solved with the stronger condition  $\mathcal{H}_0(\mathbf{x}) = 0$ <sup>6</sup> despite of the fact that these two conditions are not equivalent in general. In fact, the absence of the local form of the Hamiltonian constraint in the projectable version of Hořava-Lifshitz theory has fatal consequence for the spectrum of perturbative modes that contain either tachyon or ghost modes [29]. It is clear that the same problems occur in the gauge fixed version of projectable Hořava-Lifshitz gravity which is RFDiff invariant Hořava-Lifshitz gravity as was nicely shown in [6]. However formally the projectable Hořava-Lifshitz gravity is well defined system with the Hamiltonian given as a linear combination of the first class constraints so that it is possible to perform the gauge fixing that leads to RFDiff invariant Hořava-Lifshitz gravity.

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<sup>6</sup>For very nice analysis of this issue, see [21].

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